J. Fluid Mech. (1969), *vol.* 37, *part* 3, *pp.* 449–456
Printed in Great Britain

A law of the wall for compressible turbulent boundary layers with air injection

By L. C. SQUIRE
Cambridge University Engineering Department

(Received 13 August 1968)

A considerable body of experimental data now exists concerning turbulent boundary layers with air injection at the wall, both at subsonic and at supersonic speeds. In the present report these data for Mach numbers up to 6·5 have been analyzed to find the parameters which occur in the law of the wall as deduced from mixing-length theory. Although the absolute values of the parameters are subject to error because of the lack of accurate skin-friction measurements, the trends of these parameters with Mach number and injection mass flow are clearly defined.

1. Introduction

In the past few years there has been considerable interest in turbulent boundary layers on porous surfaces with injection of fluid through the surface. The effect of this fluid injection is to deform the velocity and temperature distributions through the boundary layer so that the skin friction and heat transfer are reduced. Furthermore, the actual process of passing cold fluid through the wall cools the wall. Thus the total process is a powerful means of cooling surfaces exposed to a hot gas stream. Much of the experimental work on this topic has concentrated on measurements of overall skin friction and heat transfer for various injection fluids. However, there are now detailed measurements for the development of velocity and temperature profiles over porous surfaces at Mach numbers up to 6·5 (see, for example, Mickley & Davies 1957, Stevenson 1964, Danberg 1964, McQuaid 1968, Jeromin 1968*a* and Squire 1968).

In addition to these experimental studies there have been a number of theoretical studies of the problem. Most of these use mixing-length theory for the fully turbulent region, but complete analysis is hampered by lack of knowledge of the various parameters which are needed to use the theory. To some extent this problem was overcome by Jeromin (1968*b*), who showed that the fully turbulent part of the compressible turbulent boundary layer with air injection could be transformed into the corresponding region of an incompressible layer. However, the problem of finding the relevant transformation parameters is still present. In this paper the experimental data are analyzed to find the parameters in the law of the wall, as obtained by straightforward application† of mixing-length theory.

† Since the layers considered here are in near equilibrium, it is to be expected that mixing-length theory will give an adequate description of the mean velocity profile (Townsend 1961).

In spite of the recent advances in numerical techniques for solving turbulent boundary-layer flows, law-of-the-wall analysis still has an important role to play since many of these numerical techniques only treat the outer part of the layer and match the numerical solution to the appropriate law of the wall.

2. Law of the wall from mixing-length theory

As mentioned in the introduction, mixing-length theory can be used to study the fully turbulent part of the compressible turbulent layer with air injection. The law of the wall can be derived from this theory as follows. The two-dimensional equations for the compressible turbulent boundary layer in zero pressure gradient are

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v) = 0, \tag{1}$$

and

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y} = \frac{\partial \tau}{\partial y}, \tag{2}$$

where $\tau = \mu(\partial u/\partial y)-\overline{\rho u'v'}$. The boundary conditions for flow over a porous surface are

$$u = 0, \quad v = v_w, \quad \text{at} \quad y = 0,$$

$$u \to U_1 \quad \text{as} \quad y \to \infty,$$

where v_w is the injection velocity. By assuming that derivatives in the x-direction are negligible compared with derivatives in the y-direction, equation (1) can be integrated to give

$$\rho v = \text{const.} = \rho_w v_w, \tag{3}$$

and with this result equation (2) can be integrated to give

$$\rho_w v_w u = \tau-\tau_w. \tag{4}$$

In the fully turbulent region, mixing-length theory (with the assumption that $l = ky$) gives

$$\tau = \rho k^2 y^2 (du/dy)^2, \tag{5}$$

so that equation (4) can be formally integrated to give

$$\int_{u_a}^{u} \frac{\sqrt{\rho}\, du'}{\sqrt{(\rho_w v_w u'+\tau_w)}} = \frac{1}{k}\ln\frac{y}{y_a}, \tag{6}$$

where u_a and y_a are the values of u and y at the inner edge of the fully turbulent region.

Equation (6) may be rearranged to give

$$\int_{0}^{u} \frac{\sqrt{\rho}\, du'}{\sqrt{(\rho_w v_w u'+\tau_w)}} = \frac{1}{k}\ln\frac{yu_\tau}{\nu_w}-\frac{1}{k}\ln\frac{y_a u_\tau}{\nu_w}+\int_{0}^{u_a} \frac{\sqrt{\rho}\, du'}{\sqrt{(\rho_w v_w u'+\tau_w)}}$$

$$= \frac{1}{k}\ln\frac{yu_\tau}{\nu_w}+B \quad \text{(say)}, \tag{7}$$

where u_τ is the friction velocity $(= \sqrt{(\tau_w/\rho_w)}$, and B is a parameter to be determined.

Before the left-hand side of equation (7) can be integrated, it is necessary to know the density. For turbulent boundary layers on solid surfaces with small heat transfer it has been shown that the static temperature, and hence the density, is related to the velocity ratio by the approximate formula

$$\frac{T}{T_w} = \frac{\rho_w}{\rho} = 1 + \frac{T_r - T_w}{T_w}\left(\frac{u}{U_1}\right) + \frac{T_1 - T_r}{T_w}\left(\frac{u}{U_1}\right)^2, \tag{8}$$

where T_w is the wall temperature, T_r is the recovery temperature and T_1 is the free-stream static temperature. Danberg (1964) and Jeromin (1968*a*) have confirmed that this relationship also holds for layers with air injection.

With this result the left-hand side of equation (7) may be written as

$$\sqrt{\left(\frac{T_1}{T_w}\right)}\int_0^\phi \left(\frac{\rho_w v_w}{\rho_1 U_1}\phi' + \tfrac{1}{2}c_f\right)^{-\frac{1}{2}}\left(1 + \frac{T_r - T_w}{T_w}\phi' + \frac{T_1 - T_r}{T_w}\phi'^2\right)^{-\frac{1}{2}} d\phi' = \Phi, \tag{9}$$

say, where $\phi = u/U_1$. Thus the law of the wall for compressible turbulent boundary layers with air injection is

$$\Phi = \frac{1}{k}\ln\frac{yu_\tau}{\nu_w} + B. \tag{10}$$

Rubesin (1954) has shown that the integral Φ can be evaluated in terms of elliptic integrals. However, in all the work described here Φ has been found by numerical integration.

For incompressible constant-temperature flows Φ reduces to

$$\int_0^\phi \left(\frac{v_w}{U_1}\phi' + \tfrac{1}{2}c_f\right)^{-\frac{1}{2}} d\phi' = \frac{2U_1}{v_w}\left\{\sqrt{\left(\frac{v_w u}{U_1^2} + \tfrac{1}{2}c_f\right)} - \sqrt{(\tfrac{1}{2}c_f)}\right\},$$

so that the law of the wall for incompressible layers with air injection can be written

$$\frac{2u_\tau}{v_w}\left\{\sqrt{\left(1 + \frac{v_w u}{u_\tau^2}\right)} - 1\right\} = \frac{1}{k}\ln\frac{yu_\tau}{\nu} + B. \tag{11}$$

This is the law of the wall obtained by Stevenson (1963), who showed from his experimental results that the parameters k and B were virtually independent of the injection velocity. However, when Danberg (1964) evaluated Φ for his experimental data at $M = 6{\cdot}5$ he found that the parameter B fell with increase in injection rate so that, for an injection rate of $\rho_w v_w/\rho_1 U_1 = 0{\cdot}0012$, B was $2{\cdot}5$ as compared with a value of 10 for the solid wall at the same Mach number and with the same ratio of wall temperature to free-stream static temperature.

Finally, it should be noted that, for the compressible, turbulent, boundary layer on a solid surface, equation (10) reduces to the law of the wall as proposed by Wilson (1950) and Fenter & Stalmach (1958).

3. Analysis of experimental data

Before looking at the experimental results for compressible layers it is instructive to study figure 1, where one of McQuaid's (1968) velocity profiles for incompressible flow with $v_w/U_1 = 0{\cdot}0017$ is plotted in terms of the co-ordinates given by equation (11). The various sets of points on this figure were obtained by using different values of c_f in the evaluation of u_τ. As will be seen all the experimental points have a straight-line portion when plotted against $\ln yu_\tau/\nu$, and the slope of this straight line is virtually independent of c_f, whereas the level of the curve, i.e. the constant B, depends critically on c_f. This dependence of B on the assumed value of the skin-friction coefficient also occurs in compressible layers and so introduces some uncertainty into the analysis of the experimental data. It is thus necessary to describe in some detail the methods used to find the skin-friction coefficients used in the present study.

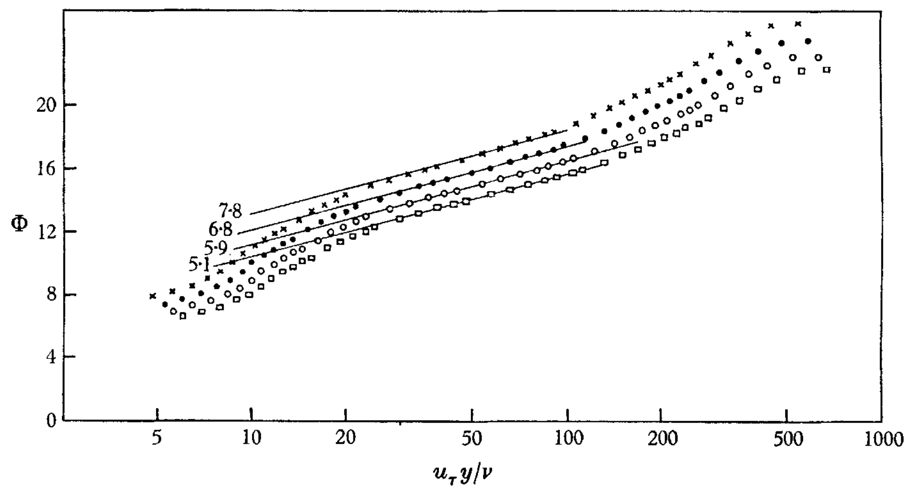

FIGURE 1. Variation of Φ with $u_\tau y/\nu$ for incompressible flow. □, $c_f = 0{\cdot}00250$; ○, $c_f = 0{\cdot}00221$; ●, $c_f = 0{\cdot}00196$; ×, $c_f = 0{\cdot}00166$.

The method used was as follows. First the values of c_f as quoted in the experimental reports were plotted in the form c_f/c_{f0} against $2F/c_{f0}$, where c_{f0} is the skin-friction coefficient in the absence of injection,† and $F\ (= \rho_w v_w/\rho_1 U_1)$ is the injection parameter. The results are shown by the points plotted in figure 2. Next these points were enclosed by scatter bands based on the quoted experimental accuracy. These bands are shown by the solid lines in figure 2. The actual variation of c_f/c_{f0} was then taken as the mean of these scatter bands (figure 3). The value of c_f for any set of data was then found by taking the appropriate value of c_f/c_{f0} from figure 3 and multiplying this ratio by a value of c_{f0} found from the correlation presented by Spalding & Chi (1964). In general the values of c_{f0} found from Spalding & Chi agreed with the quoted values to within the quoted experimental accuracy; similar agreement was found for the values of c_f.

Using the values of c_f obtained by the above method the experimental results of Jeromin (1968a) for a Mach number of 3·5, and of Squire (1968) for Mach numbers of 1·8 and 2·5 were plotted in the form of Φ against $\ln(u_\tau y/\nu)$. For all these experimental results the wall temperature is slightly higher than the recovery temperature $((T_r - T_w)/T_r \approx -0{\cdot}06)$. Figures 4 and 5 show typical results for one station at Mach numbers of 1·8 and 3·5. For zero injection, the parameter B has a value of about 4·8 for all three Mach numbers. This value is consistent with the results quoted by Danberg, who shows that, for zero injection, the

† This form of plot was used since it appears to collapse results at different Reynolds numbers.

parameter B increases for heat transfer to the wall, and decreases for heat transfer from the wall. Similar curves to those shown in figures 4 and 5 were obtained by Danberg for injection at Mach numbers near 6·5 for two rates of heat transfer $((T_r - T_w)/T_r \approx 0{\cdot}1$ and $0{\cdot}4)$. All the experimental curves plotted in figures 4 and 5 have a clearly defined linear portion, and the slope of the linear

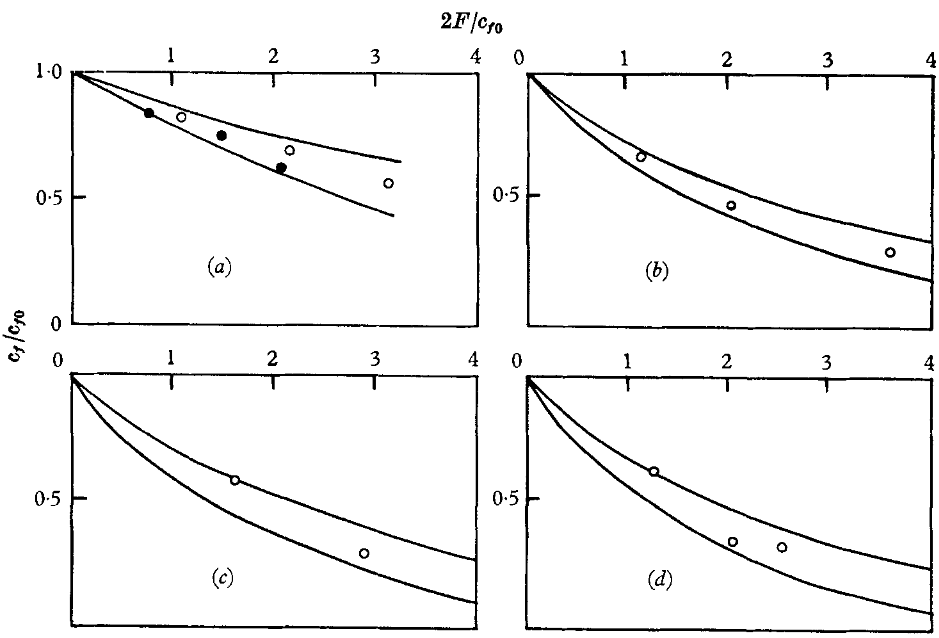

FIGURE 2. Experimental variation of c_f/c_{f0} with $2F/c_{f0}$. (a) $M = 6{\cdot}3$, (b) $M = 3{\cdot}55$, (c) $M = 2{\cdot}5$, (d) $M = 1{\cdot}8$. ○, $T_w/T_1 \approx 4$; ●, $T_w/T_1 \approx 7{\cdot}6$, for $M = 6{\cdot}3$.

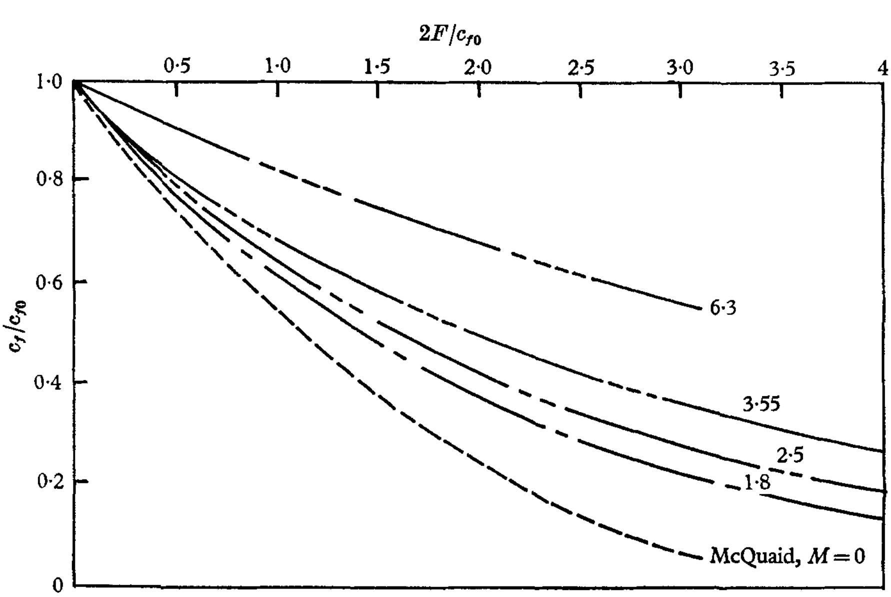

FIGURE 3. Mean variation of c_f/c_{f0}.

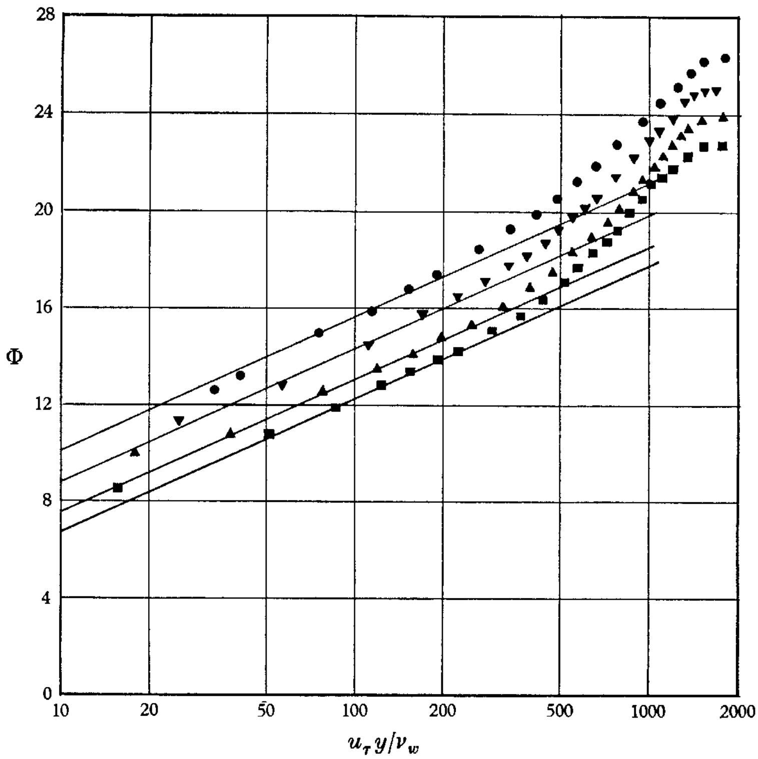

FIGURE 4. Variation of Φ with $u_\tau y/\nu_w$ at $M = 1{\cdot}8$. ●, $F = 0$; ▼, $F = 0{\cdot}0012$; ▲, $F = 0{\cdot}0025$; ■, $F = 0{\cdot}0031$.

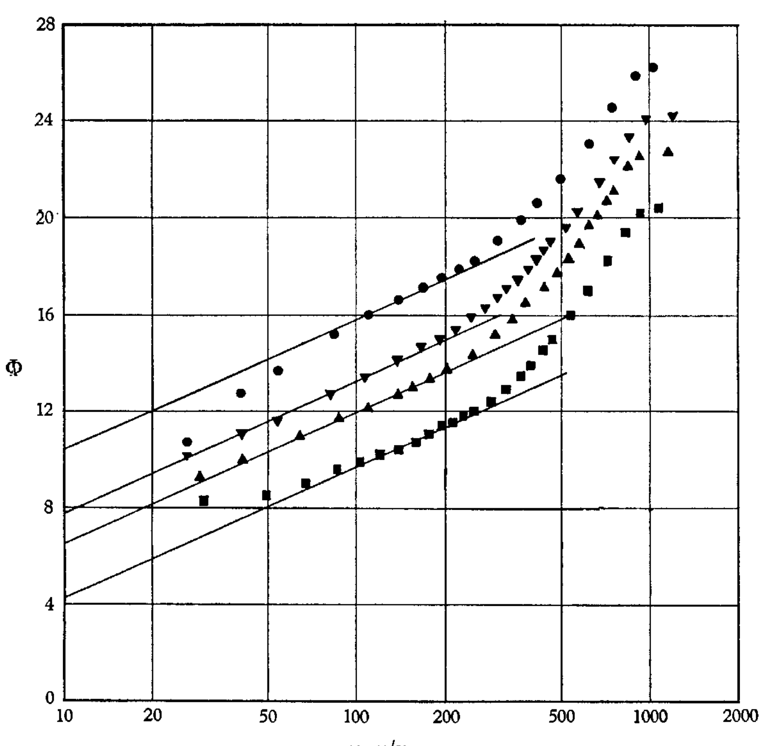

FIGURE 5. Variation of Φ with $u_\tau y/\nu_w$ at $M = 3{\cdot}55$. ●, $F = 0$; ▼, $F = 0{\cdot}00065$; ▲, $F = 0{\cdot}0012$; ■, $F = 0{\cdot}0021$.

region is virtually independent of Mach number and injection rate. In fact all the lines drawn through this region have a slope corresponding to a value of k of 0·42, i.e. a value of the mixing-length constant well within the range obtained for incompressible flow. When the experimental results were re-plotted using

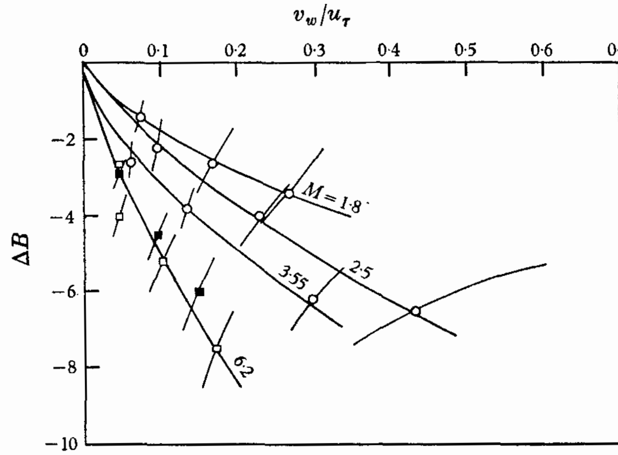

FIGURE 6. Variation of ΔB with v_w/u_τ. ■, $(T_r - T_w)/T_r \approx 0{\cdot}1$; □, $(T_r - T_w) T_w \approx 0{\cdot}4$; ○, see text below.

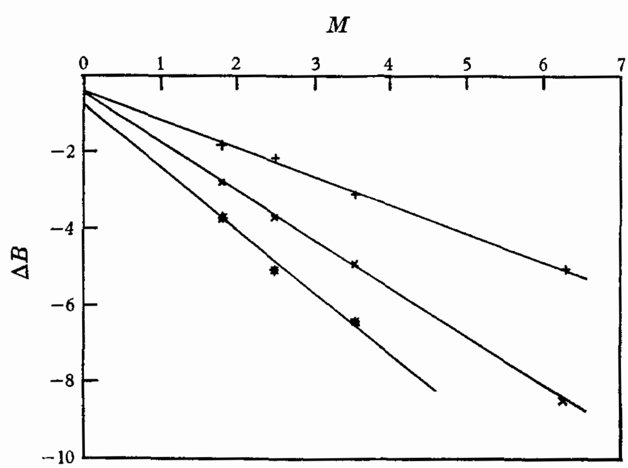

FIGURE 7. Variation of ΔB with mean Mach number. +, $v_w/u_\tau = 0{\cdot}1$; ×, $v_w/u_\tau = 0{\cdot}2$; ✱, $v_w/u_\tau = 0{\cdot}3$.

other values of skin-friction coefficient, there was no change in the slope of the linear part of the curve, but the level did change, as it did for the results plotted in figure 1. Results for all the experimental profiles are summarized in figure 6, where ΔB is plotted against v_w/u_τ. Here ΔB is the difference between the experimental value of B at a given Mach number and injection rate and the experimental value of B at the same Mach number with no injection, i.e.

$$\Delta B = B(F, M) - B(F = 0, M).$$

In figure 6 the points correspond to the values of ΔB obtained using the skin-friction coefficients found from the mean curves of figure 3, while the lines through the points show the scatter which arises from using skin-friction coefficients within the bands of figure 2. In spite of this possible scatter, it is clear that the magnitude of ΔB increases with Mach number at fixed v_w/u_τ. Further, figure 7 shows that when the values of ΔB based on the mean skin-friction coefficients are plotted against Mach number at fixed v_w/u_τ the points lie on straight lines which all pass close to the origin. Thus figures 6 and 7 appear to support Stevenson's suggestion that B is independent of injection in incompressible flow. However, as shown in a recent paper by Stevenson (1968) the actual variation of B with v_w/u_τ in incompressible flow is still uncertain since it depends critically on the measured variation of skin-friction coefficient with injection, and a number of workers have found different forms of this variation, and hence different variations of B with v_w/u_τ. Stevenson suggests that the different forms of the variation of skin friction with injection may be associated with the nature of the porous surface and this is a problem which requires further investigation.

Finally, it is of interest to note that Danberg's results show virtually no effect of heat transfer on the variations of B with v_w/u_τ (see figure 6).

4. Conclusions

The experimental results presented in this paper show that, for air injection into a turbulent boundary layer, a clearly defined law of the wall exists; and, further, that the mixing-length constant k is independent of Mach number and injection rate. However, the additive parameter in the law varies with both Mach number and injection, and the value depends critically on the measured skin-friction coefficient. In general, this additive parameter falls with increasing injection rate at fixed Mach number, and the rate of fall increases with increase in Mach number. The present results suggest that in incompressible flow the additional parameter is almost independent of injection.

REFERENCES

Danberg, J. E. 1964 *NOL T.R.* 64–99, *Aero. Res. Rep.* no. 228.
Fenter, F. W. & Stalmach, C. J. 1958 *J. Aero/Space Sci.* **25**, 793.
Jeromin, L. O. F. 1968*a* *Aero. Res. Coun. R & M* 3526.
Jeromin, L. O. F. 1968*b* *J. Fluid Mech.* **31**, 65.
McQuaid, J. 1968 *Aero. Res. Coun. R & M* 3549.
Mickley, H. S. & Davies, R. S. 1957 *NACA Tech. Note* no. 4017.
Rubesin, M. W. 1954 *NACA Tech. Note* no. 3341.
Spalding, D. B. & Chi, S. W. 1964 *J. Fluid Mech.* **18**, 117.
Squire, L. C. 1968 To be published.
Stevenson, T. N. 1963 *College of Aeronautics (Cranfield) Rep. Aero.* 166.
Stevenson, T. N. 1964 *College of Aeronautics (Cranfield) Rep. Aero.* 177.
Stevenson, T. N. 1968 *AIAA J.* **6**, 553.
Townsend, A. A. 1961 *J. Fluid Mech.* **11**, 97.
Wilson, R. E. 1950 *J. Aero. Sci.* **17**, 585.